Efficient identification of sparse neural networks with butterfly structure

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BUTTERFLY FACTORIZATION

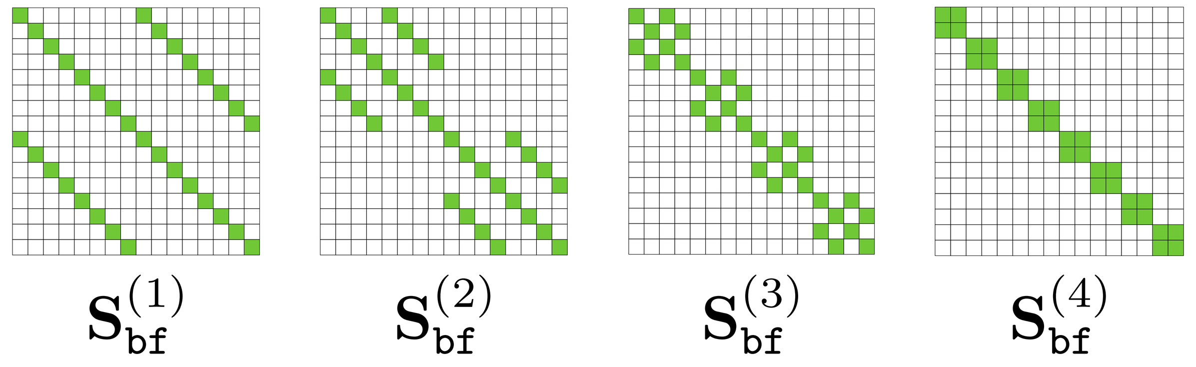
Given $\mathbf{Z} \in \mathbb{C}^{N \times N}$ with $N = 2^J$:

$$\min_{\mathbf{X}^{(1)},\ldots,\mathbf{X}^{(J)}} \|\mathbf{Z} - \mathbf{X}^{(1)} \ldots \mathbf{X}^{(J)}\|_F^2$$

such that $\mathrm{supp}(\mathbf{X}^{(\ell)}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(\ell)} := \mathbf{I}_{2^{\ell-1}} \otimes \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \otimes \mathbf{I}_{N/2^{\ell}}$

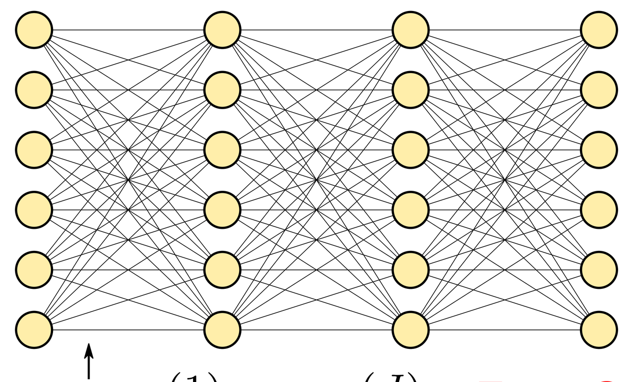
where $\mathrm{supp}(\mathbf{M}) := \{(i,j) \mid \mathbf{M}_{i,j} \neq 0\}$.

Example with J=4 factors:



MOTIVATIONS

REPLACE DENSE MATRICES BY BUTTERFLY MATRICES



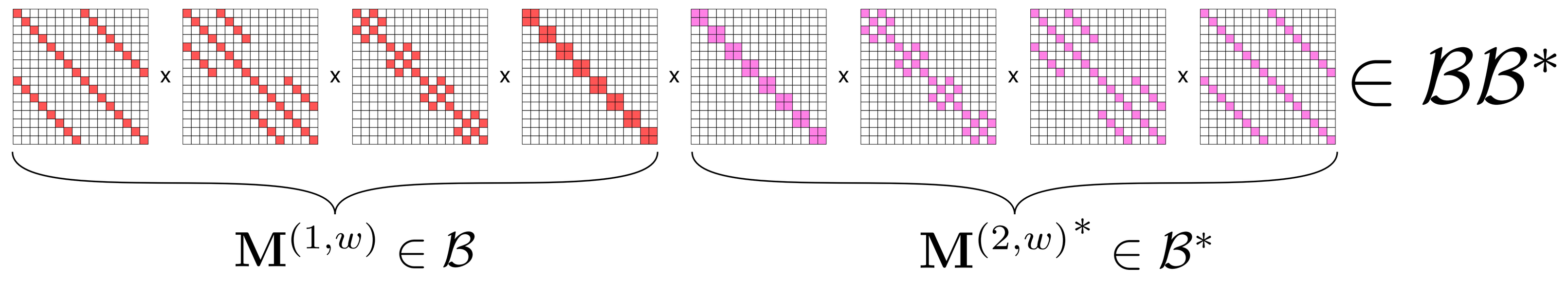
Less parameters
Faster training / inference time

$\mathbf{W} \approx \mathbf{X}^{(1)} \ldots \mathbf{X}^{(J)}$ Fast O(N log N) matrix-vector multiplications

THE BUTTERFLY STRUCTURE IS EXPRESSIVE

$$\mathcal{B} := \{\mathbf{X}^{(1)} \ldots \mathbf{X}^{(J)} \mid \mathrm{supp}(\mathbf{X}^{(\ell)}) \subseteq \mathrm{supp}(\mathbf{S}_{\mathtt{bf}}^{(\ell)})\}$$

Kaleidoscope representation [2]:



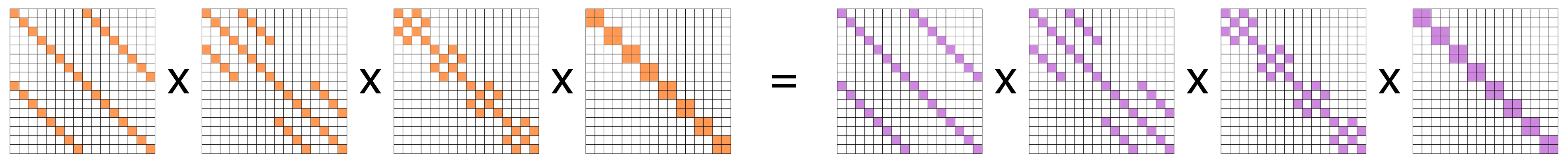
$\prod_{w=1}^{W} \mathbf{M}^{(1,w)}\mathbf{M}^{(2,w)*}$ captures

DFT, DSC, DCT, Hadamard
Convolutions, Toeplitz
Orthogonal random features
Fastfood, ACDC

CONTRIBUTIONS

IDENTIFIABILITY

If $\mathbf{Z} = \mathbf{X}^{(1)} \times \mathbf{X}^{(2)} \times \ldots \times \mathbf{X}^{(J)} = \bar{\mathbf{X}}^{(1)} \times \bar{\mathbf{X}}^{(2)} \times \ldots \times \bar{\mathbf{X}}^{(J)}$

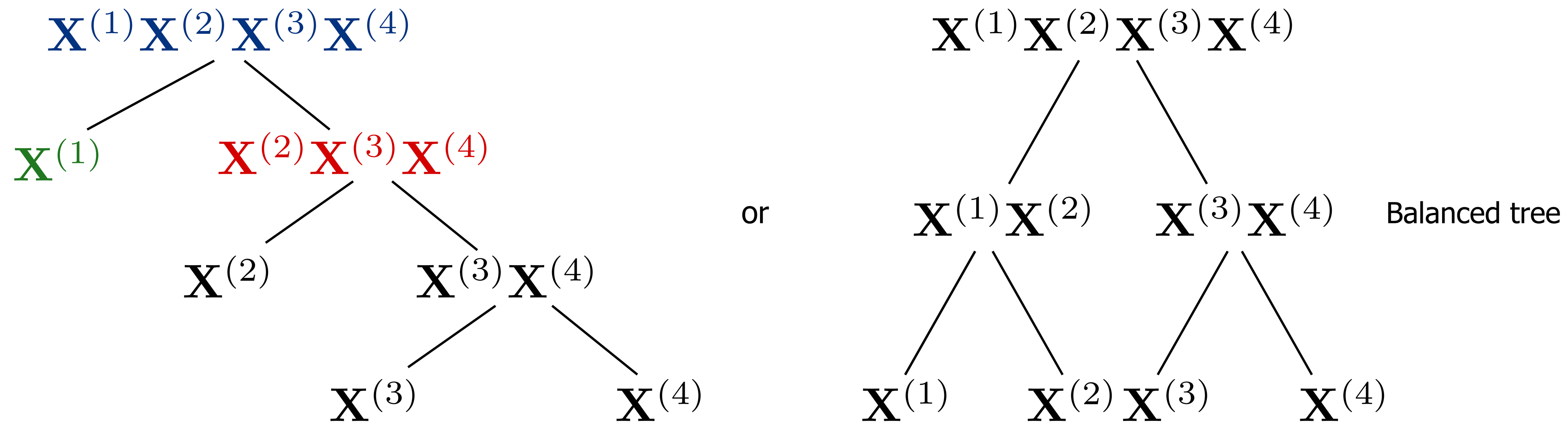


then $(\mathbf{X}^{(\ell)})_{\ell=1}^{J} \sim (\bar{\mathbf{X}}^{(\ell)})_{\ell=1}^{J}$ up to natural scaling ambiguity.

HIERARCHICAL ALGORITHM

Let $\mathbf{Z} := \mathbf{X}^{(1)} \ldots \mathbf{X}^{(J)}$ with exact butterfly factorization.

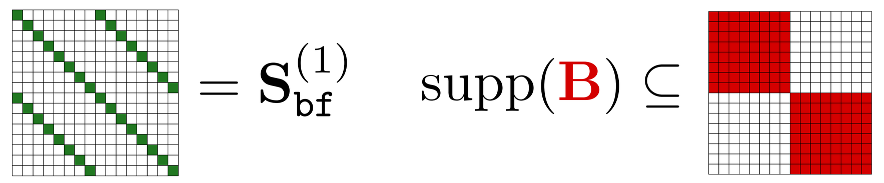
Principle: perform successive two-layer fixed-support factorizations, until obtaining J factors.



Exact recovery guarantee: the butterfly factors are recovered from $\mathbf{Z}$ by the algorithm, in O(N²) time.

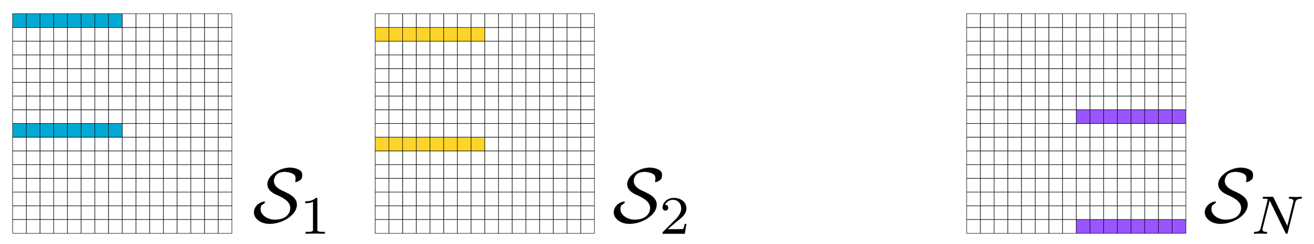
AT THE FIRST LEVEL of the unbalanced tree:

(FSMF) $\min_{\mathbf{A},\mathbf{B}} \|\mathbf{Z} - \mathbf{A}\mathbf{B}\|_F^2$ such that $\mathrm{supp}(\mathbf{A}) \subseteq$ [pattern] $= \mathbf{S}_{\mathtt{bf}}^{(1)}$ $\mathrm{supp}(\mathbf{B}) \subseteq$ [pattern] $= \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)}$



Decomposition into rank-one matrices: $\mathbf{A}\mathbf{B} = \mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet} + \mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet} + \ldots + \mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}$

Observation: the rank-one supports are pairwise disjoint.



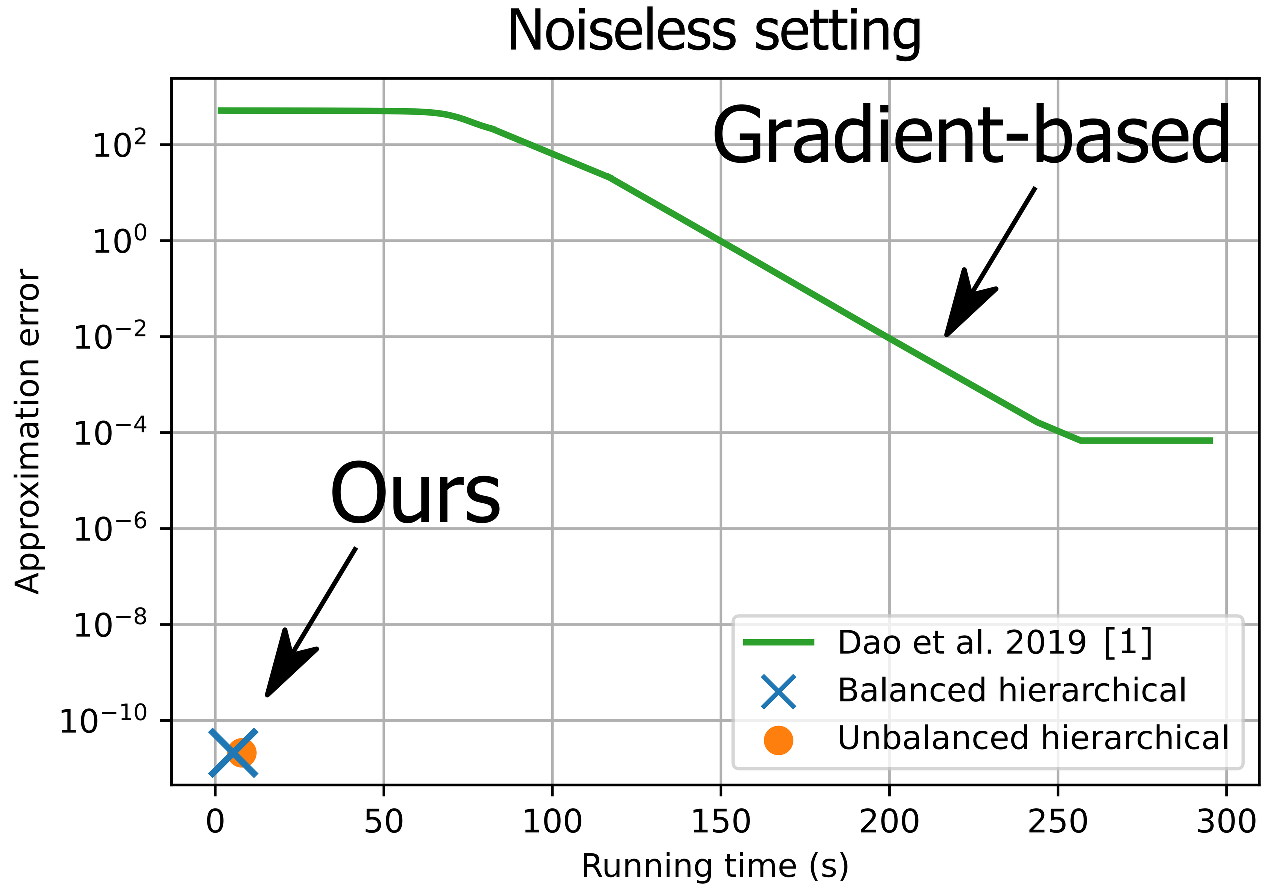
THEOREM:
(FSMF) is polynomially solvable [3] and admits an essentially unique solution [5].

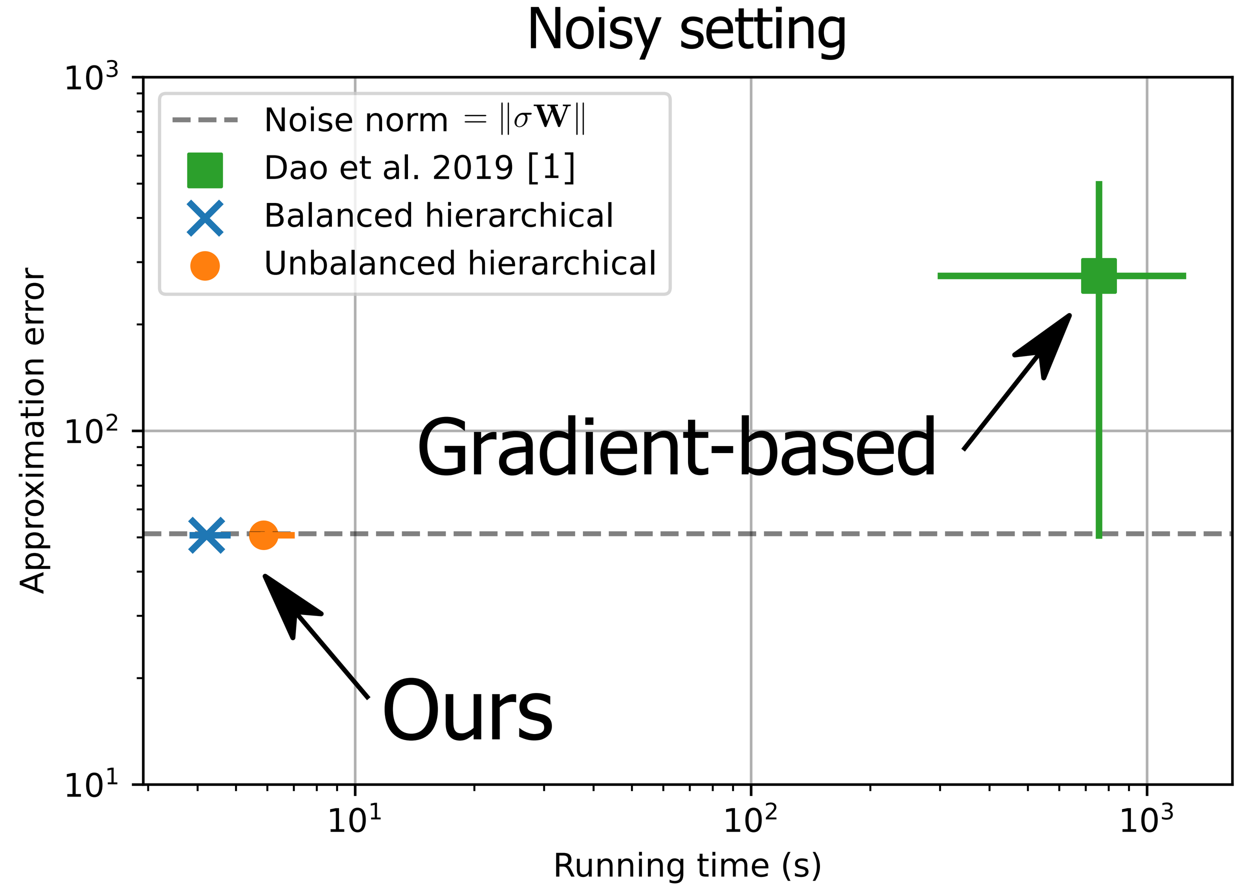
ALGORITHM to solve (FSMF) [3]:

1. Extract the submatrices $\mathbf{Z} \odot \mathcal{S}_i$ for $i = 1, \ldots, N$
2. Perform best rank-one approximation for each submatrix

NUMERICAL EXPERIMENTS

Recovery of the butterfly factors of the DFT matrix:





Experiments are taken from [4].

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