Efficient Identification of Butterfly Sparse Matrix Factorizations

Léon Zheng, Elisa Riccietti, Rémi Gribonval

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Sparse matrix factorization

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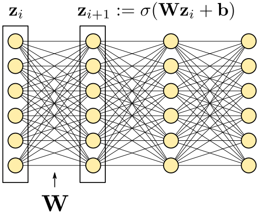

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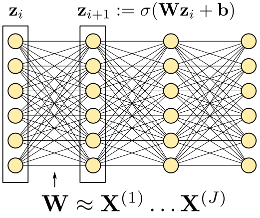

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When is the problem well-posed? Uniqueness of solution? Stability?
$\rightarrow$ Still an open question.

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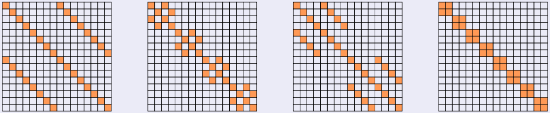

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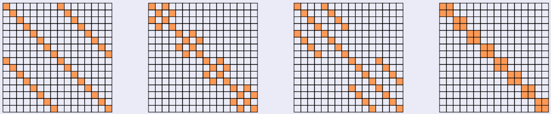

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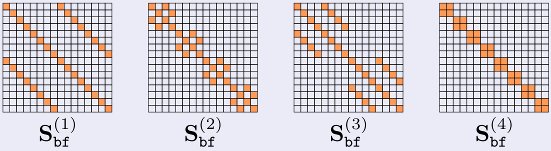

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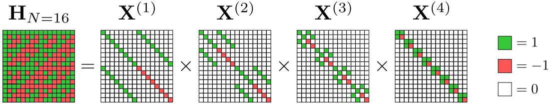

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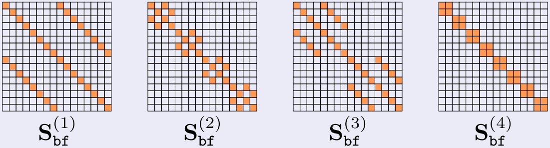

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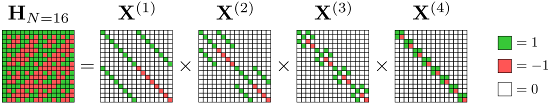

3. Makes the sparse matrix factorization problem well-posed
   [L. Zheng et al., **Efficient identification of butterfly sparse matrix factorizations**, SIMODS, 2023]

Overview of this talk

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2. Factors are recovered by a flexible hierarchical factorization algorithm.
3. The algo. is numerically faster, more accurate than gradient-descent.

Hierarchical factorization algorithm

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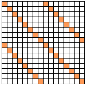
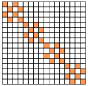
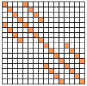
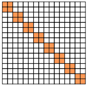
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 

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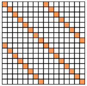
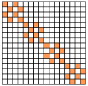
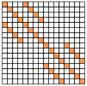
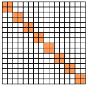
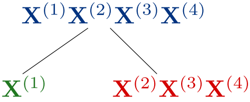
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



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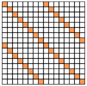
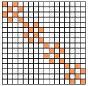
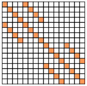
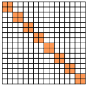
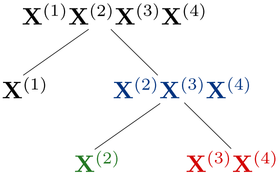
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



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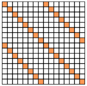
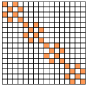
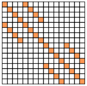
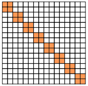
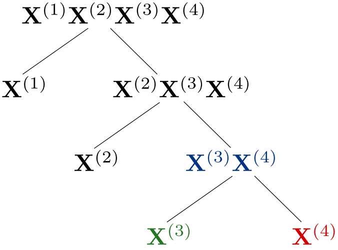
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



Hierarchical factorization algorithm

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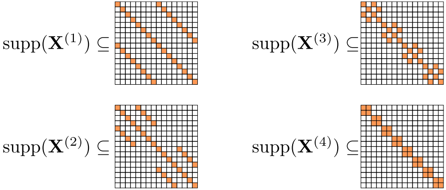

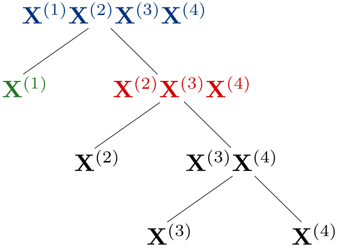

How to recover the partial products?

Hierarchical factorization algorithm

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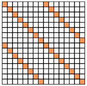
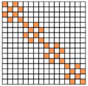
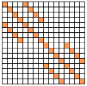
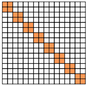
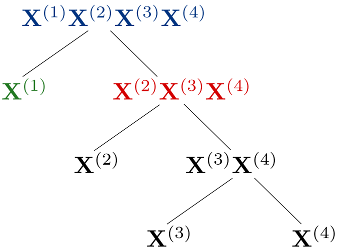
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

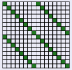
$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

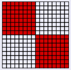
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



How to recover the partial products? $\rightarrow$ use their known supports

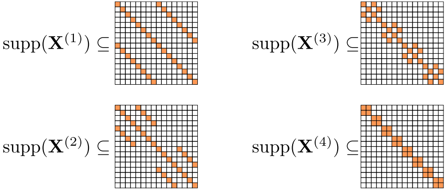
Lemma (Supports of the partial products)

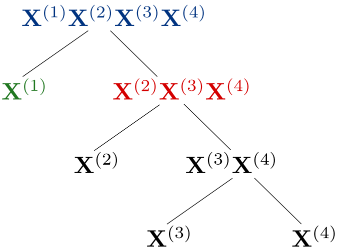
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(1)}$

$\mathrm{supp}(\mathbf{X}^{(2)}\mathbf{X}^{(3)}\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)}$

Hierarchical factorization algorithm

Let $\mathbf{Z} := \mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)}\mathbf{X}^{(4)}$ such that:





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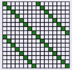
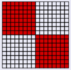
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(1)}$

$\mathrm{supp}(\mathbf{X}^{(2)}\mathbf{X}^{(3)}\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)}$

Two-layer fixed-support problem:

$$\min_{\mathbf{A},\mathbf{B}} \|\mathbf{Z} - \mathbf{A}\mathbf{B}\|_F \,, \text{ s.t. } \mathrm{supp}(\mathbf{A}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(1)}, \ \mathrm{supp}(\mathbf{B}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)} \qquad (\star)$$

Two-layer fixed-support sparse matrix factorization

$$\min_{\mathbf{A},\mathbf{B}} \|\mathbf{Z} - \mathbf{A}\mathbf{B}\|_F \,, \text{ s.t.} \operatorname{supp}(\mathbf{A}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(1)}, \operatorname{supp}(\mathbf{B}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)} \qquad (\star)$$

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Fact: $\mathbf{AB} = \sum_{i=1}^{N} \mathbf{A}_{\bullet,i}\mathbf{B}_{i,\bullet}$.

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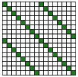
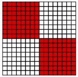
Constraint on the pair of factors

$\operatorname{supp}(\mathbf{A}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(1)}$

$\operatorname{supp}(\mathbf{B}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)}$

Two-layer fixed-support sparse matrix factorization

$$\min_{\mathbf{A},\mathbf{B}} \|\mathbf{Z} - \mathbf{A}\mathbf{B}\|_F \,, \text{ s.t. } \operatorname{supp}(\mathbf{A}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(1)}, \ \operatorname{supp}(\mathbf{B}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)} \qquad (\star)$$

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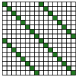
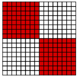
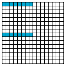
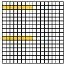
$\operatorname{supp}(\mathbf{A}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(1)}$

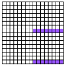
$\operatorname{supp}(\mathbf{B}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)}$

Constraint on the rank-one matrices

$\operatorname{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

$\operatorname{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

$\vdots$

$\operatorname{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

Two-layer fixed-support sparse matrix factorization

$$\min_{\mathbf{A},\mathbf{B}} \|\mathbf{Z} - \mathbf{A}\mathbf{B}\|_F \,, \text{ s.t. } \mathrm{supp}(\mathbf{A}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(1)}, \; \mathrm{supp}(\mathbf{B}) \subseteq \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)} \qquad (\star)$$

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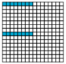
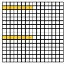
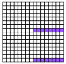
$\mathrm{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

$\mathrm{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

$\cdots$

$\mathrm{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

Theorem ([Le et al. 2022; Zheng et al. 2023])

The rank-one matrices have **pairwise disjoint supports**.

Two-layer fixed-support sparse matrix factorization

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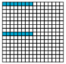
$\operatorname{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

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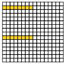
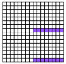
$\operatorname{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

$\operatorname{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

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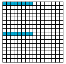
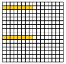
$\mathrm{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

$\mathrm{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

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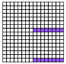
$\mathrm{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

Theorem ([Le et al. 2022; Zheng et al. 2023])

The rank-one matrices have **pairwise disjoint supports***. Consequently:*

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2. *the solution is essentially unique in the noiseless setting.*

Two-layer fixed-support sparse matrix factorization

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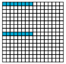
$\mathrm{supp}(\mathbf{A}_{\bullet,1}\mathbf{B}_{1,\bullet}) \subseteq$  $= \mathcal{S}_1$

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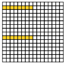
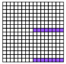
$\mathrm{supp}(\mathbf{A}_{\bullet,2}\mathbf{B}_{2,\bullet}) \subseteq$  $= \mathcal{S}_2$

$\mathrm{supp}(\mathbf{A}_{\bullet,N}\mathbf{B}_{N,\bullet}) \subseteq$  $= \mathcal{S}_N$

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The rank-one matrices have **pairwise disjoint supports***. Consequently:*

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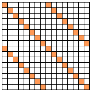
2. *the solution is essentially unique in the noiseless setting.*

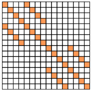
Algorithm:

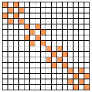
1. Extract the submatrices $\mathbf{Z}_{|\mathcal{S}_i}$, $i = 1, \ldots, N$
2. Perform best rank-one approximation for each submatrix

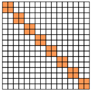
Hierarchical factorization algorithm

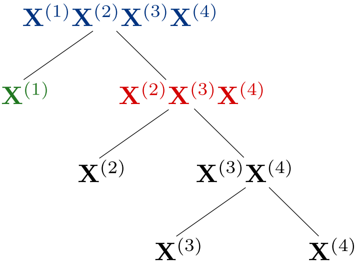
Let $\mathbf{Z} := \mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)}\mathbf{X}^{(4)}$ such that:

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

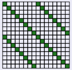
$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

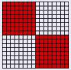
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



The two-layer procedure is repeated **recursively**.

Lemma (Support of the partial products)

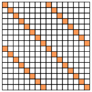
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(1)}$

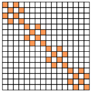
$\mathrm{supp}(\mathbf{X}^{(2)}\mathbf{X}^{(3)}\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}\mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)}$

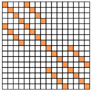
Corresponding rank-one supports are pairwise disjoint.

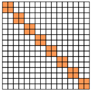
Hierarchical factorization algorithm

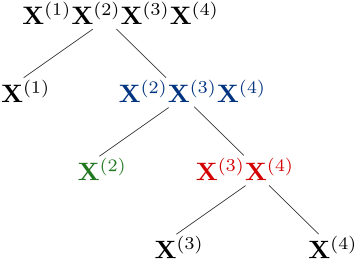
Let $\mathbf{Z} := \mathbf{X}^{(1)}\mathbf{X}^{(2)}\mathbf{X}^{(3)}\mathbf{X}^{(4)}$ such that:

$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

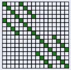
$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

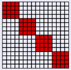
$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



The two-layer procedure is repeated **recursively**.

Lemma (Support of the partial products)

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(2)}$

$\mathrm{supp}(\mathbf{X}^{(3)}\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}\mathbf{S}_{\mathtt{bf}}^{(4)}$

Corresponding rank-one supports are pairwise disjoint.

Hierarchical factorization algorithm

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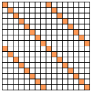
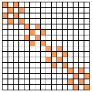
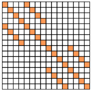
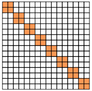
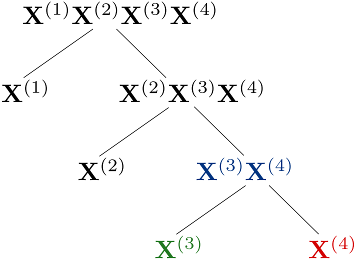
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



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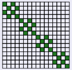
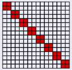
$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}$

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(4)}$

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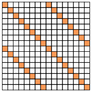
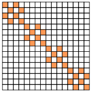
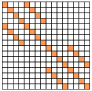
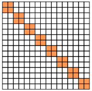
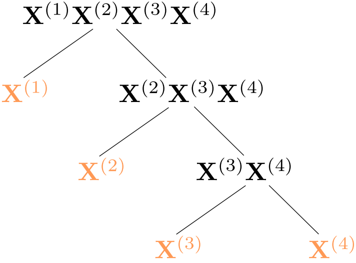
$\mathrm{supp}(\mathbf{X}^{(1)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(2)}) \subseteq$ 

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$ 



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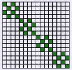
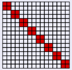
$\mathrm{supp}(\mathbf{X}^{(3)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(3)}$

$\mathrm{supp}(\mathbf{X}^{(4)}) \subseteq$  $= \mathbf{S}_{\mathtt{bf}}^{(4)}$

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$\{\mathbf{X}^{(\ell)}\}_{\ell=1}^{4}$ are recovered from $\mathbf{Z}$, up to scaling ambiguities.

Uniqueness of butterfly factorization

Theorem ([Zheng et al. 2023])

Except for trivial degeneracies, the butterfly factorization $\mathbf{Z} := \mathbf{X}^{(1)} \dots \mathbf{X}^{(J)}$ *is essentially unique, up to unavoidable scaling ambiguities:*

$$\begin{cases} \bar{\mathbf{X}}^{(1)} \dots \bar{\mathbf{X}}^{(J)} = \mathbf{Z} \\ \forall \ell \in [J], \ \mathrm{supp}(\bar{\mathbf{X}}^{(\ell)}) \subseteq \mathbf{S}_{\mathrm{bf}}^{(\ell)} \end{cases} \quad \Longrightarrow \ (\bar{\mathbf{X}}^{(\ell)})_{\ell=1}^{J} \sim (\mathbf{X}^{(\ell)})_{\ell=1}^{J}$$

The unique factors are recovered from the hierarchical algorithm.

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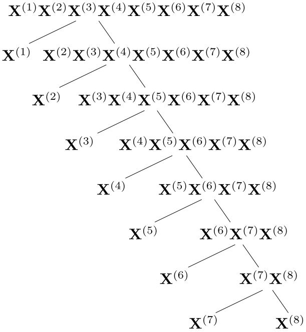
Proof: at level ℓ, the intermediate matrix to factorize is

$$\mathbf{M} := \underbrace{(\mathbf{D}^{-1}\mathbf{X}^{(\ell)})}_{\mathbf{A}} \underbrace{(\mathbf{X}^{(\ell+1)} \dots \mathbf{X}^{(J)} \tilde{\mathbf{D}})}_{\mathbf{B}}.$$

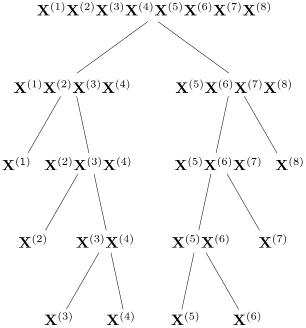
The algorithm recovers $(\mathbf{A}, \mathbf{B})$ because of optimality & uniqueness in $(\star)$.

Flexible choice in the hierarchical order

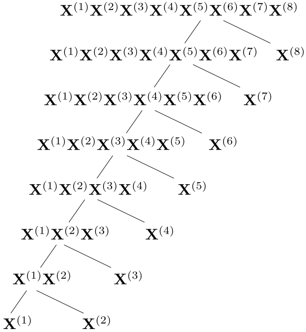
The algorithm works for any factor-bracketing binary tree.



Column-wise factorization



Hybrid factorization



Row-wise factorization

This extends existing work that consider only 3 trees.

[Y. Liu et al., **Butterfly factorization via randomized matrix-vector multiplications**, SISC, 2021]

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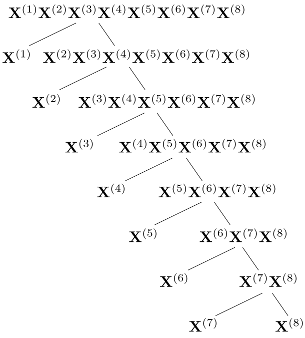
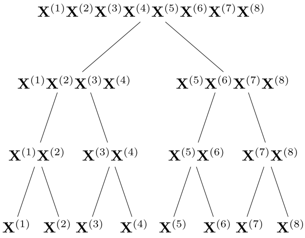


Unbalanced tree



Balanced tree (new)

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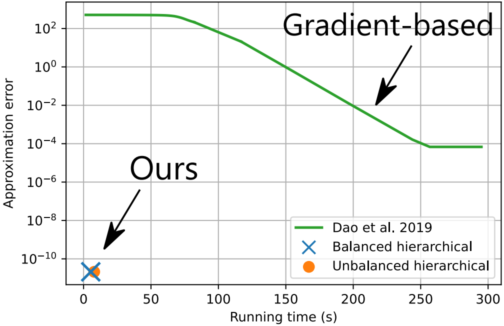
Faster and more accurate in the noiseless setting

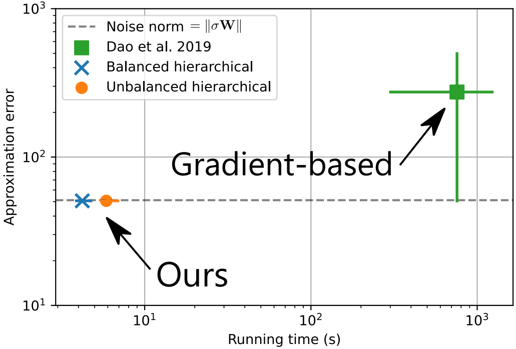
Approximate $\mathbf{Z} := \mathbf{DFT}_{512}$ by a product of $J = 9$ butterfly factors:



The theoretical complexity of the algorithm is $\mathcal{O}(N^2)$.

Also more robust in the noisy setting

Approximate $\mathbf{Z} := \mathbf{DFT}_{512} + \sigma \mathbf{W}$ by a product of $J = 9$ butterfly factors:



FAµST Toolbox: faust.inria.fr

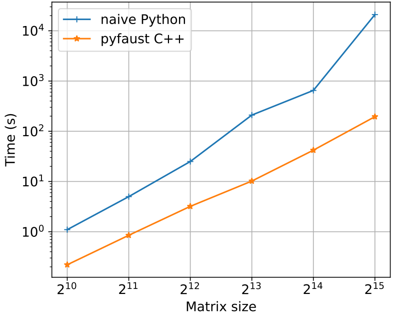
Efficient implem. of fast transforms and algorithms for sparse matrix fact.

- Python & MATLAB wrappers (C++ core, GPU compatible)
- PYPI install: `pip install pyfaust`

Ex: butterfly factorization

```
from pyfaust.fact import butterfly
from pyfaust import Faust, wht

H = wht(8).toarray()
F = butterfly(H, type='bbtree')
(F-H).norm() / Faust(H).norm()
# Output: 1.2560739454502295e-15
```



Factorization of the Hadamard matrix with a balanced tree.

Conclusion and perspectives

1. The butterfly structure captures many common fast transforms.
2. We proved the essential uniqueness of the butterfly factorization.
3. Butterfly factors are recovered by a flexible hierarchical algorithm.

Hierarchical algorithm: $\mathcal{O}(N^2)$

$\mathbf{Z} \in \mathbb{R}^{N \times N}$ (dense)	$\tilde{\mathbf{Z}} := \mathbf{X}^{(J)}\mathbf{X}^{(J-1)} \ldots \mathbf{X}^{(1)}$
Storage: $\mathcal{O}(N^2)$	Storage: $\mathcal{O}(N \log N)$
Cost for evaluation: $\mathcal{O}(N^2)$ $\mathbf{x} \mapsto \mathbf{Z}\mathbf{x}$	Cost for evaluation: $\mathcal{O}(N \log N)$ $\mathbf{x} \mapsto \tilde{\mathbf{Z}}\mathbf{x}$

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On going work

- Approximation error of the hierarchical algorithm
- Taking into account row and column permutations
- Efficient training of neural networks with butterfly structure

Thank you for your attention!

To know more:

Q.-T. Le, L. Zheng, E. Riccietti, and R. Gribonval
Fast learning of fast transforms, with guarantees
In *ICASSP*, 2022.

Q.-T. Le, E. Riccietti, and R. Gribonval
Spurious valleys, NP-hardness, and tractability of sparse matrix factorization with fixed support
In *SIAM Journal on Matrix Analysis and Applications*, 2022.

L. Zheng, E. Riccietti, and R. Gribonval
Efficient identification of butterfly sparse matrix factorizations
In *SIAM Journal on Mathematics of Data Science*, 2023.