Identifiability in matrix sparse factorization

Léon Zheng

leon.zheng@ens-lyon.fr

M2 Internship, Inria DANTE, LIP (ENS de Lyon)

Supervisor: Rémi Gribonval (Inria DANTE / LIP)

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INVENTEURS DU MONDE NUMÉRIQUE

Overview

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Motivation: algorithm for matrix sparse factorization

Given a matrix $\boldsymbol{Z}$, we want to find some sparse factors $(\boldsymbol{X}_\ell)_{\ell=1}^{L}$ such that:

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Optimization problem

Let $\boldsymbol{Z}$ be an observed matrix, and $(\mathcal{E}_\ell)_{\ell=1}^L$ some sparsity constraint sets. We want to solve [Le Magoarou et al., 2016]:

$$\underset{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_L}{\text{Minimize}} \quad \underbrace{\|\boldsymbol{Z} - \prod_{\ell=1}^{L} \boldsymbol{X}_\ell\|^2}_{\text{Data-fidelity}} \quad + \quad \underbrace{\sum_{\ell=1}^{L} g_{\mathcal{E}_\ell}(\boldsymbol{X}_\ell)}_{\text{Sparsity-inducing penalty}} \quad . \tag{1}$$

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Difficulties:

- Nonconvex optimization
- Combinatorial issues

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In matrix sparse factorization, what are the conditions which guarantee successful recovery of the sparse factors?

- This is still an open question.
- It leads to the question of identifiability, which is about the uniqueness of the sparse factors in the recovery.

Analogy with linear sparse recovery [Foucart et al., 2017]

Linear sparse recovery problem

Recover a signal $\boldsymbol{x} \in \mathbb{C}^N$ from an observed data $\boldsymbol{y} \in \mathbb{C}^m$, given the linear model:

$$\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}.$$

Sparsity assumption on the signal $\boldsymbol{x}$: allows reconstruction when $m < N$.

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Conditions for the success of these algorithms?

Conditions for which the signal $\boldsymbol{x}$ is identifiable, *i.e.*, it is the unique solution of the sparse recovery problem, when we observe $\boldsymbol{y} = \boldsymbol{A}\boldsymbol{x}$?

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→ Identifiability is well studied for linear inverse problems [Foucart et al., 2017], but not for multilinear inverse problems, like matrix sparse factorization.

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Let $\boldsymbol{Z} \in \mathbb{C}^{n \times m}$ be a matrix. Consider the bilinear inverse problem:

$$\begin{aligned} \text{find} \quad & (\boldsymbol{X}, \boldsymbol{Y}) \\ \text{such that} \quad & \boldsymbol{X}\boldsymbol{Y} = \boldsymbol{Z}, \\ & \boldsymbol{X}, \boldsymbol{Y} \text{ are } \underline{\text{sparse}}. \end{aligned} \tag{2}$$

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- Sparsity: a matrix is sparse if its support is *allowed*. We choose what are the allowed supports.
- Equivalence relations: scaling + permutations, because

$$\boldsymbol{X}\boldsymbol{Y} = (\boldsymbol{X}\boldsymbol{D})(\boldsymbol{D}^{-1}\boldsymbol{Y}) = (\boldsymbol{X}\boldsymbol{P})(\boldsymbol{P}^T\boldsymbol{Y})$$

  where $\boldsymbol{D}$ is a diagonal matrix, and $\boldsymbol{P}$ is a permutation matrix.

Contributions

1. Characterization of fixed-support identifiability
2. Characterization of right identifiability

We observe $\boldsymbol{Z} := \boldsymbol{X}\boldsymbol{Y}$.

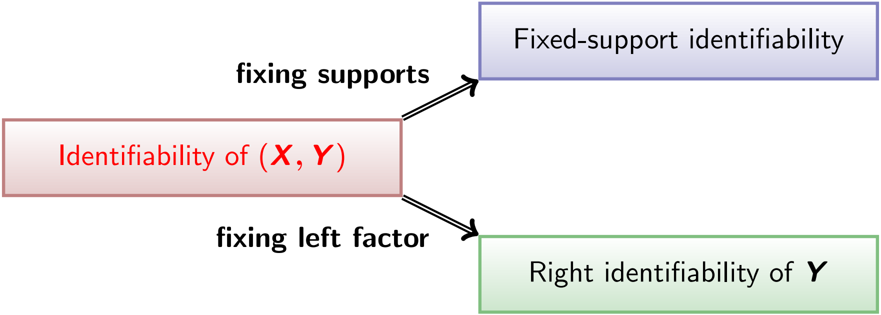

Figure: Deriving necessary conditions of identifiability by considering two problem variations

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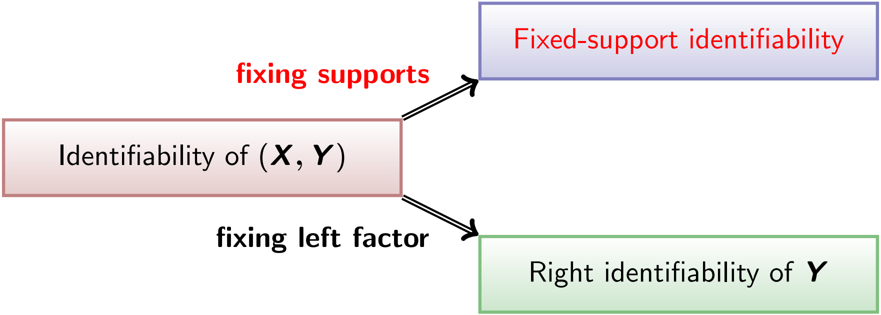

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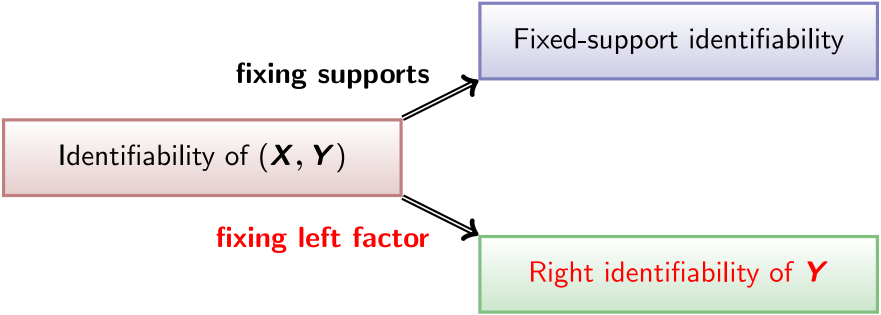

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Overview

Fixed-support identifiability definition

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Example:

$$(\boldsymbol{S_X}, \boldsymbol{S_Y}) := \left(\begin{pmatrix} \boxed{\star \;\; \star} \\ 0 \;\; 0 \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\star} \\ 0 & \boxed{\star} \end{pmatrix} \right)$$

$$(\boldsymbol{X_1}, \boldsymbol{Y_1}) := \left(\begin{pmatrix} \boxed{\mathbf{1} \;\; \mathbf{2}} \\ 0 \;\; 0 \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\mathbf{3}} \\ 0 & \boxed{\mathbf{4}} \end{pmatrix} \right)$$

(a) Allowed supports

$$(\boldsymbol{X_2}, \boldsymbol{Y_2}) := \left(\begin{pmatrix} \boxed{\mathbf{2} \;\; 0} \\ 0 \;\; 0 \end{pmatrix}, \begin{pmatrix} 0 & \boxed{\mathbf{2}} \\ \mathbf{1} & \boxed{\mathbf{1}} \end{pmatrix} \right)$$

(b) Not allowed supports

Fixed-support identifiability definition

Consider $(\boldsymbol{S_X}, \boldsymbol{S_Y})$ a fixed pair of supports.

<u>Example</u>:

$$(\boldsymbol{S_X}, \boldsymbol{S_Y}) := \left(\begin{array}{cc} \boxed{\star \quad \star} \\ 0 \quad 0 \end{array} \right), \left(\begin{array}{c|c} 0 & \star \\ 0 & \star \end{array} \right)$$

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(b) Not allowed supports

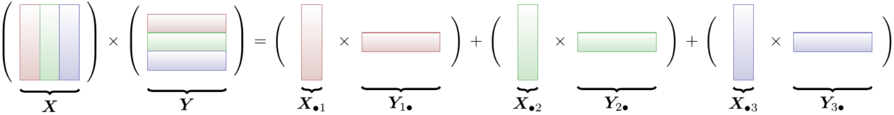
Definition: identifiability of $(\boldsymbol{S_X}, \boldsymbol{S_Y})$

Every pair $(\boldsymbol{X}, \boldsymbol{Y})$ with a support equal to $(\boldsymbol{S_X}, \boldsymbol{S_Y})$ is the unique solution (up to equivalence) for the factorization of $\boldsymbol{Z} := \boldsymbol{XY}$ into two factors supported by $(\boldsymbol{S_X}, \boldsymbol{S_Y})$.

$\rightarrow$ We will give here a characterization of this property.

Rank 1 contributions representation

Let $(\boldsymbol{X}, \boldsymbol{Y})$ be a pair of factor.



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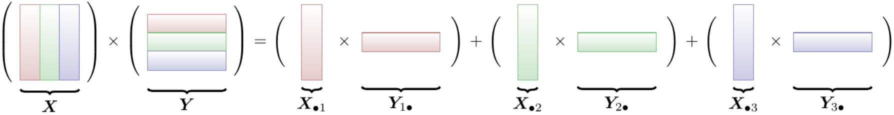

Definition

$(\boldsymbol{X}, \boldsymbol{Y})$ is represented by $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^{r}$, where r is the number of columns in $\boldsymbol{X}$ (or rows in $\boldsymbol{Y}$).

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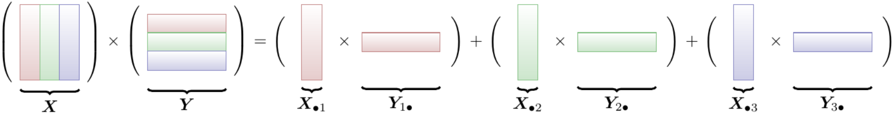

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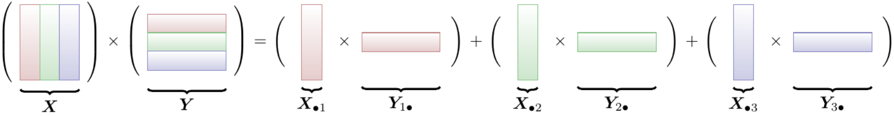

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$\rightarrow$ [Le Magoarou, 2016] used this representation to show that the butterfly factorization of the Discrete Fourier Transform matrix is identifiable.

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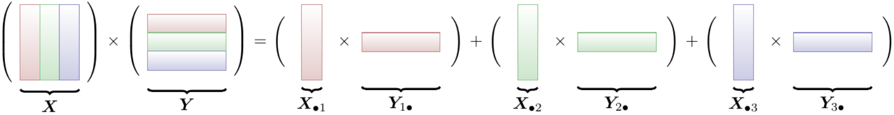

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$\rightarrow$ We are implicitly using lifting ideas, inspired by [Choudhary et al., 2014], [Malgouyres et al., 2016]. The lifting operator is $\mathscr{S} : (\boldsymbol{C_i})_{i=1}^{r} \mapsto \sum_{i=1}^{r} \boldsymbol{C_i}$.

Identifiability of the rank 1 contributions?

We now observe $\boldsymbol{Z} := \boldsymbol{X}\boldsymbol{Y}$.

Identifiability of $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^{r}$ from the observation $\boldsymbol{Z}$?

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$$\begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{2} \\ 0 & \mathbf{2} & 0 & \mathbf{4} \\ 0 & \mathbf{3} & 0 & \mathbf{6} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{2} & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{2} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{5} & \mathbf{5} & \mathbf{6} \\ 0 & 0 & \mathbf{3} & 0 \end{pmatrix}$$

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Idea

Complete each rank 1 contribution from the entries not covered by the other rank 1 contributions.

Example

We know: the observed matrix $\boldsymbol{Z}$, and the supports of the rank 1 contributions $((\boldsymbol{S_X})_{\bullet i}(\boldsymbol{S_Y})_{i\bullet})_{i=1}^{r}$.
We want: to reconstruct the rank 1 contributions $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^{r}$.

$$\boldsymbol{Z} = \begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{2} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{5} & \mathbf{5} & \mathbf{6} \\ 0 & 0 & \mathbf{3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & ? & 0 & ? \\ 0 & ? & 0 & ? \\ 0 & ? & 0 & ? \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & ? & 0 \\ 0 & 0 & ? & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ ? & ? & ? & 0 \\ ? & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure: How to reconstruct the rank 1 contributions?

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Figure: We show in color the "observable" entries. The red contribution is completable from its observable entries.

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$$= \begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{2} \\ 0 & \mathbf{2} & 0 & \mathbf{4} \\ 0 & \mathbf{3} & 0 & \mathbf{6} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & ? & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & ? & \mathbf{3} & 0 \\ \mathbf{1} & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure: This “uncovers” entries in the green contribution.

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Figure: Now it is possible to complete the green contribution.

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We know: the observed matrix $\boldsymbol{Z}$, and the supports of the rank 1 contributions $((\boldsymbol{S_X})_{\bullet i}(\boldsymbol{S_Y})_{i\bullet})_{i=1}^{r}$.
We want: to reconstruct the rank 1 contributions $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^{r}$.

$$\begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{2} \\ \mathbf{1} & \mathbf{4} & \mathbf{3} & \mathbf{4} \\ \mathbf{1} & \mathbf{5} & \mathbf{5} & \mathbf{6} \\ 0 & 0 & \mathbf{3} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{2} \\ 0 & ? & 0 & \mathbf{4} \\ 0 & ? & 0 & \mathbf{6} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & ? & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & ? & \mathbf{3} & 0 \\ \mathbf{1} & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & \mathbf{1} & 0 & \mathbf{2} \\ 0 & \mathbf{2} & 0 & \mathbf{4} \\ 0 & \mathbf{3} & 0 & \mathbf{6} \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \mathbf{2} & 0 \\ 0 & 0 & \mathbf{3} & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{3} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure: Therefore, $(\boldsymbol{X}_{\bullet i}\boldsymbol{Y}_{i\bullet})_{i=1}^{r}$ are identifiable from the observation $\boldsymbol{Z}$.

Iterative completability from observable supports

Let $\boldsymbol{S}$ be a rank 1 support (= support of a rank 1 matrix).

Definition: $\boldsymbol{S}$ is completable from $\boldsymbol{S}' \subseteq \boldsymbol{S}$

We can complete any rank 1 matrix $\boldsymbol{M}$ with a support equal to $\boldsymbol{S}$, by observing only its entries on $\boldsymbol{S}'$.

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Let $\boldsymbol{S_1}, \ldots, \boldsymbol{S_r}$ be r rank 1 supports.

Definition: iterative completability of $(\boldsymbol{S_i})_{i=1}^r$

- The rank 1 supports $\boldsymbol{S_i}$ for $i \in [\![1; r]\!]$ can be completed one by one from its observable support:

$$\boldsymbol{S_i} \setminus \bigcup_{i' \in [\![r]\!] \setminus \{i\}} \boldsymbol{S_{i'}}.$$

- When the i-th rank 1 support is completable from its observable support, we repeat with $(\boldsymbol{S_{i'}})_{i \neq i'}$.

Iterative completability from observable supports

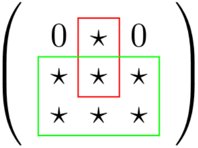


Figure: This example is iteratively completable.

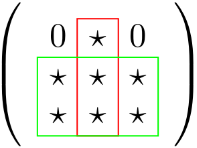


Figure: This example is not iteratively completable.

Fixed-support identifiability characterization

Theorem

For $r = 2$, $(\boldsymbol{S_X}, \boldsymbol{S_Y})$ is identifiable if, and only if, the supports of its rank 1 contributions are iteratively completable.

Remark: Sufficiency is true for all r.

Fixed-support identifiability characterization

Theorem

For $r = 2$, $(\boldsymbol{S_X}, \boldsymbol{S_Y})$ is identifiable if, and only if, the supports of its rank 1 contributions are iteratively completable.

Remark: Sufficiency is true for all r. Necessity is false for $r \geq 3$.

$$\begin{pmatrix} 0 & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \star & ? & 0 & 0 \\ \star & ? & 0 & 0 \\ \star & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \star & \star & 0 \\ 0 & ? & \star & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & ? & \star \\ 0 & ? & \star & \star \end{pmatrix}$$

Figure: Counterexample showing that iterative completability is not a necessary condition for fixed-support identifiability.

$\rightarrow$ This leads to the notion of iterative partial completability (future work).

Overview

Some right identifiability results

Consider $\boldsymbol{X}$ a fixed left factor, and Ω_R a family of allowed right supports.

Theorem

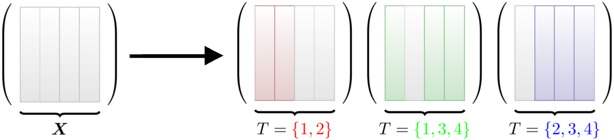
Suppose that $\boldsymbol{X}$ non-degenerate, and Ω_R is stable by inclusion. Then the following assertions are equivalent:

1. Ω_R *is right identifiable for* $\boldsymbol{X}$;
2. *the columns of* $\boldsymbol{X}$ *indexed by* T *are linearly independent, for all*

$$T \in \mathcal{T}(\Omega_R)$$

where $\mathcal{T}(\Omega_R)$ *is a collection of indices subsets determined by* Ω_R.

Example: for a specific Ω_R, we can have $\mathcal{T} = \{\{1,2\},\{1,3,4\},\{2,3,4\}\}$.



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Example (Family of right supports l-sparse by row)

Condition: all the columns of $\boldsymbol{X}$ are linearly independent.

Some right identifiability results

Consider $\boldsymbol{X}$ a fixed left factor, and Ω_R a family of allowed right supports.

Theorem

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Example (Family of right supports k-sparse by column)

Condition: every subset of $2k$ columns of $\boldsymbol{X}$ is linearly independent.

$\rightarrow$ Similar result in compressive sensing literature [Foucart et al., 2017].

Overview

Conclusion

Summary

1. Fixed-support identifiability: with **rank 1 matrix completion** conditions.
2. Right identifiability: with **linear independence** of specific subsets of columns in the left factor.

Conclusion

Summary

1. Fixed-support identifiability: with **rank 1 matrix completion** conditions.
2. Right identifiability: with **linear independence** of specific subsets of columns in the left factor.

Open questions

- Fixed-support identifiability: characterization with **iterative partial completability**?
- Finding sufficient conditions of **generic identifiability**? Necessary and sufficient conditions?

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Extra: existing identifiability results

1. **Lifting** for identifiability in generic bilinear inverse problems [Choudhary et al., 2014]

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Given a bilinear mapping $\boldsymbol{S} : (\boldsymbol{x}, \boldsymbol{y}) \mapsto \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{y})$, derive $\mathscr{S} : \boldsymbol{W} \mapsto \mathscr{S}(\boldsymbol{W})$, with the identity: $\mathscr{S}(\boldsymbol{x}\boldsymbol{y}^T) = \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{y})$. Then:

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$$\begin{aligned} &\text{find} && (\boldsymbol{x}, \boldsymbol{y}) \\ &\text{such that} && \boldsymbol{S}(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{z}, \\ & && (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{K}. \end{aligned} \quad \Longleftrightarrow \quad \begin{aligned} &\text{minimize} && \text{rank}(\boldsymbol{W}) \\ &\text{such that} && \mathscr{S}(\boldsymbol{W}) = \boldsymbol{z}, \\ & && \boldsymbol{W} \in \mathcal{K}'. \end{aligned}$$

where $\mathcal{K}' \cap \{\text{matrix } \boldsymbol{W} \text{ with rank at most } 1\} = \{\boldsymbol{x}\boldsymbol{y}^T \mid (\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{K}\}$.

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Proposition (Identifiability characterization [Choudhary et al., 2014])

$\text{Ker}\, \mathscr{S} \cap \{\textit{matrix } \boldsymbol{W} \textit{ with rank at most } 2\} \cap (\mathcal{K}' - \mathcal{K}') = \{0\}$.

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Notation: $\omega = exp(i\frac{2\pi}{N})$. Here, for instance, $N = 4$.

$$\underbrace{\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & \omega^1 & \omega^2 & \omega^3 \\ 1 & \omega^2 & 1 & \omega^2 \\ 1 & \omega^3 & \omega^2 & \omega^1 \end{pmatrix}}_{\text{DFT matrix } N \times N} = \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & \omega^1 \\ 1 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & \omega^3 \end{pmatrix}}_{\frac{N}{2}\text{-sparse by column}} \underbrace{\begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & \omega^2 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & \omega^2 \end{pmatrix}}_{\text{2-sparse by row}}$$

Left support: $\frac{N}{2}$-sparse by column. Right support: 2-sparse by row.

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Rank 1 matrix completability:

$$\boldsymbol{M} = \begin{pmatrix} 0 & \star & ? & ? \\ 0 & \star & ? & ? \\ 0 & \star & \star & \star \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Figure: Can we complete missing entries (?) from observable entries ($\star$)? The rank of $\boldsymbol{M}$ is at most 1.

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Main issue

No general conditions easy to verify for identifiability in matrix sparse factorization.

Extra: equivalence relation? sparsity?

Equivalent pairs of factors

$(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{A}, \boldsymbol{B})$ if $\boldsymbol{X}\boldsymbol{P}\boldsymbol{D} = \boldsymbol{A}$ and $\boldsymbol{D}^{-1}\boldsymbol{P}^T\boldsymbol{Y} = \boldsymbol{B}$, with:

- $\boldsymbol{D}$ a scaling matrix (diagonal, nonzero diagonal entries);
- $\boldsymbol{P}$ a permutation matrix.

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Equivalent pairs of factors

$(\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{A}, \boldsymbol{B})$ if $\boldsymbol{XPD} = \boldsymbol{A}$ and $\boldsymbol{D}^{-1}\boldsymbol{P}^T\boldsymbol{Y} = \boldsymbol{B}$, with:

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Family of allowed supports

Let Ω be a subset of supports. $\boldsymbol{M} \in \mathbb{C}^{p \times q}$ is sparse $\iff$ $\mathrm{supp}(\boldsymbol{M}) \in \Omega$.

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Support of a matrix $\boldsymbol{M} \in \mathbb{C}^{p \times q}$ as a binary matrix

Denote $\text{supp}(\boldsymbol{M}) \in \{0, 1\}^{p \times q}$ where $\text{supp}(\boldsymbol{M})_{ij} = 1 \iff \boldsymbol{M}_{ij} \neq 0$.

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Family of allowed pairs of supports

Let $\hat{\Omega}$ be a subset of pairs of supports. $(\boldsymbol{X}, \boldsymbol{Y}) \in \mathbb{C}^{n \times r} \times \mathbb{C}^{r \times m}$ is sparse $\iff$ $(\text{supp}(\boldsymbol{X}), \text{supp}(\boldsymbol{Y})) \in \hat{\Omega}$.

Extra: definition of identifiability

Consider $\hat{\Omega}$ a family of allowed pairs of supports.

Definition: identifiability of $\hat{\Omega}$

For all $(\boldsymbol{X}, \boldsymbol{Y}), (\boldsymbol{A}, \boldsymbol{B})$ with allowed support in Ω, we have:

$$\boldsymbol{X}\boldsymbol{Y} = \boldsymbol{A}\boldsymbol{B} \Rightarrow (\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{A}, \boldsymbol{B}).$$

Problem formulation: under which condition $\hat{\Omega}$ is identifiable?

Extra: right identifiability is a necessary condition

Given $\hat{\Omega}$ a family of allowed pairs of supports, and $\boldsymbol{X}$ a left factor, denote:

$$\Omega_R(\boldsymbol{X}) := \{\boldsymbol{S_Y} \mid (\text{supp}(\boldsymbol{X}), \boldsymbol{S_Y}) \in \hat{\Omega}\}.$$

Extra: right identifiability is a necessary condition

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Lemma

If $\hat{\Omega}$ is identifiable, then for all left factors $\boldsymbol{X}$, $\Omega_R(\boldsymbol{X})$ is right identifiable for $\boldsymbol{X}$.

Extra: right identifiability is a necessary condition

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If $\hat{\Omega}$ is identifiable, then for all left factors $\boldsymbol{X}$, $\Omega_R(\boldsymbol{X})$ is right identifiable for $\boldsymbol{X}$.

Definition: right identifiability of $\Omega_R(\boldsymbol{X})$ for $\boldsymbol{X}$

For all $\boldsymbol{Y}, \boldsymbol{B}$ with allowed support in $\Omega_R(\boldsymbol{X})$, we have:

$$\boldsymbol{XY} = \boldsymbol{XB} \Rightarrow (\boldsymbol{X}, \boldsymbol{Y}) \sim (\boldsymbol{X}, \boldsymbol{B}).$$

Extra: lifting principle

Lifting operator:

$$\mathscr{S} : (\boldsymbol{X_i})_{i=1}^{r} \mapsto \sum_{i=1}^{r} \boldsymbol{X_i}$$

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Proposition

$(\boldsymbol{S_X}, \boldsymbol{S_Y})$ *is identifiable if, and only if,*

Extra: lifting principle

Lifting operator:

$$\mathscr{S} : (\boldsymbol{X_i})_{i=1}^r \mapsto \sum_{i=1}^r \boldsymbol{X_i}$$

Proposition

$(\boldsymbol{S_X}, \boldsymbol{S_Y})$ is identifiable if, and only if,

$$\mathrm{Ker}(\mathscr{S}) \cap \prod_{i=1}^r (\Sigma_{\boldsymbol{S_i},1} - \Sigma_{\boldsymbol{S_i},1}) = \{0\}, \tag{3}$$

where $\boldsymbol{S_i} := (\boldsymbol{S_X})_{\bullet i}(\boldsymbol{S_Y})_{i\bullet}$ is the i-th rank 1 support of $(\boldsymbol{S_X}, \boldsymbol{S_Y})$, and:

$$\Sigma_{\boldsymbol{S_i},1} := \{\textit{matrix with rank at most 1, with a support equal to } \boldsymbol{S_i}\}.$$

Extra: iterative partial completability

Counterexample

Iterative completability is not a necessary condition for fixed-support identifiability when $r \geq 3$.

$$\begin{pmatrix} 0 & \star & \star & 0 \\ \star & \star & \star & 0 \\ \star & \star & \star & \star \\ \star & \star & \star & \star \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \star & ? & 0 & 0 \\ \star & ? & 0 & 0 \\ \star & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \star & \star & 0 \\ 0 & ? & \star & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & ? & \star \\ 0 & ? & \star & \star \end{pmatrix}$$

Figure: This example is not iteratively completable from observable supports.

Counterexample

Iterative completability is not a necessary condition for fixed-support identifiability when $r \geq 3$.

$$\begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{2} & 0 \\ \mathbf{2} & \mathbf{6} & \mathbf{5} & \mathbf{6} \\ \mathbf{3} & \mathbf{5} & \mathbf{2} & \mathbf{4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & ? & 0 & 0 \\ \mathbf{2} & ? & 0 & 0 \\ \mathbf{3} & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & ? & \mathbf{2} & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & ? & \mathbf{6} \\ 0 & ? & \mathbf{2} & \mathbf{4} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & ? & 0 & 0 \\ \mathbf{2} & ? & 0 & 0 \\ \mathbf{3} & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & \mathbf{3} & \mathbf{6} \\ 0 & ? & \mathbf{2} & \mathbf{4} \end{pmatrix}$$

Figure: However, we can complete partially green and blue contributions.

Extra: iterative partial completability

Counterexample

Iterative completability is not a necessary condition for fixed-support identifiability when $r \geq 3$.

$$\begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} & 0 \\ \mathbf{1} & \mathbf{2} & \mathbf{2} & 0 \\ \mathbf{2} & \mathbf{6} & \mathbf{5} & \mathbf{6} \\ \mathbf{3} & \mathbf{5} & \mathbf{2} & \mathbf{4} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & ? & 0 & 0 \\ \mathbf{2} & ? & 0 & 0 \\ \mathbf{3} & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & ? & \mathbf{2} & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & ? & \mathbf{6} \\ 0 & ? & \mathbf{2} & \mathbf{4} \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ \mathbf{1} & \mathbf{1} & 0 & 0 \\ \mathbf{2} & ? & 0 & 0 \\ \mathbf{3} & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & \mathbf{1} & \mathbf{2} & 0 \\ 0 & ? & \mathbf{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & \mathbf{3} & \mathbf{6} \\ 0 & ? & \mathbf{2} & \mathbf{4} \end{pmatrix}$$

Figure: This "uncovers" entries in red and green contributions.

Extra: iterative partial completability

Counterexample

Iterative completability is not a necessary condition for fixed-support identifiability when $r \geq 3$.

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 6 & 5 & 6 \\ 3 & 5 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & ? & 0 & 0 \\ 2 & ? & 0 & 0 \\ 3 & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & ? & 2 & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & ? & 6 \\ 0 & ? & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & 3 & 6 \\ 0 & ? & 2 & 4 \end{pmatrix}$$

Figure: Then, red and green contributions are completable.

Extra: iterative partial completability

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Iterative completability is not a necessary condition for fixed-support identifiability when $r \geq 3$.

$$\begin{pmatrix} 0 & 1 & 2 & 0 \\ 1 & 2 & 2 & 0 \\ 2 & 6 & 5 & 6 \\ 3 & 5 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & ? & 0 & 0 \\ 2 & ? & 0 & 0 \\ 3 & ? & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & ? & 2 & 0 \\ 0 & ? & ? & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & ? & ? & 6 \\ 0 & ? & 2 & 4 \end{pmatrix}$$

$$= \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 2 & 2 & 0 & 0 \\ 3 & 3 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 3 & 3 & 6 \\ 0 & 2 & 2 & 4 \end{pmatrix}$$

Figure: We finally complete blue contribution.